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# SOME $H^\infty$-INTERPOLATING SEQUENCES AND THE BEHAVIOR OF CERTAIN OF THEIR BLASCHKE PRODUCTS

BY

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ABSTRACT. Let $f$ be a strictly increasing continuous real function defined near $0^+$ with $f(0) = f'(0) = 0$. Such a function is called a $K$-function if for every constant $k$, $f(\theta + kf(\theta))/f(\theta) \longrightarrow 1/$ as $\theta \longrightarrow 0^+$. The curve in the open unit disc with corresponding representation $1 - r = f(\theta)$ is called a $K$-curve. Several analytic and geometric conditions are obtained for $K$-curves and $K$-functions. This provides a framework for some rather explicit results involving parts in the closure of $K$-curves, $H^\infty$-interpolating sequences lying on $K$-curves and the behavior of their Blaschke products. In addition, a sequence of points in the disc tending upper tangentially to 1 with moduli increasing strictly to 1 and arguments decreasing strictly to 0 is proved to be interpolating if and only if the hyperbolic distance between successive points remains bounded away from zero.

1. **Introduction and preliminaries.** Let $D$ be the open unit disc in the complex plane. We assume that the reader is somewhat familiar with the theory of $H^\infty(D)$ as a function algebra including such basic concepts (see [1]) as its maximal ideal space, the fiber $\mathcal{D}_1$ above 1, the pseudohyperbolic metric $\chi(z, w) = |z - w|/|1 - \bar{z}w|$, and the papers [2] and [3] on the parts of $H^\infty$.

Following through the proofs of [3] it is easy to see that *all* the results there hold for a wider class of curves than those hypothesized. In particular the only property of the convex curves in that paper which was used was that these curves were $K$-curves in the sense of the following definition.

DEFINITION 1.1. Let $f$ be a strictly increasing continuous real function defined near $0^+$ with $f(0) = f'(0) = 0$. Then, $f$ is called a *K-function* if for every $k \in (-\infty, \infty)$,

$$\lim_{\theta \to 0^+} \frac{f(\theta + kf(\theta))}{f(\theta)} = 1.$$

If $\Gamma$ is an upper tangential curve in $D$ terminating at 1 whose polar

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representation is $1 - r = f(\theta)$, then $\Gamma$ is called a *K-curve*.

In §2 we will present a number of equivalent analytic and geometric criteria for $K$-curves and $K$-functions. This will lead to a new description of the various Wermer maps onto the parts in the closure of $K$-curves. We also show that the class of $K$-curves is definitely wider than the class of curves in [3].

A theorem of Wortman in [4] states that a sequence $\{z_n\}$ of points lying on a convex upper tangential curve in $D$ terminating at 1 is an interpolating sequence if and only if the numbers $\chi(z_n, z_{n+1})$ are bounded away from zero. We call a sequence $\{z_n = r_n e^{i\theta_n}\}$ in $D$ tending to 1 a (*strict*) *M-sequence* if $r_n$ increases (*strictly*) to 1, $\theta_n$ decreases (strictly) to zero and $\theta_n/(1 - r_n) \longrightarrow \infty$ as $n \longrightarrow \infty$. In §3 we prove that an $M$-sequence is interpolating if and only if it satisfies Wortman's condition, thus subsuming his result. In the special case that the points $\{z_n\}$ are (asymptotically) equally $\chi$-spaced, we study the boundary behavior of the associated Blaschke product $B$. We prove that the restriction of $B$ to any part hit by the closure of its sequence of zeros is again a Blaschke product. In fact, we explicitly compute which Blaschke product it is. One interesting topological consequence is that for any $K$-curve $\Gamma$ the set $\mathcal{D}_1 - (\Gamma^- - \Gamma)$ (the weak* closure) is disconnected.

2. **Descriptions of $K$-curves.** As in [3] we will often be interested in the quantities ($z = re^{i\theta}$, $w = \rho e^{i\varphi}$)

$$a(z) = \frac{1-r}{1+2r\cos\theta + r^2} = \mathrm{Re}\left(\frac{1-z}{1+z}\right),$$

$$b(z) = \frac{2r\sin\theta}{1+2r\cos\theta + r^2} = -\mathrm{Im}\left(\frac{1-z}{1+z}\right),$$

$$1 - \chi^2(z, w) = \frac{(1-r^2)(1-\rho^2)}{1 - 2r\rho\cos(\theta - \varphi) + r^2\rho^2}.$$

One simple result we will soon refer to is

LEMMA 2.1. *Let $\{re^{i\theta}\}$ and $\{\rho e^{i\varphi}\}$ be two nets in D tending to 1 indexed by the same index set. If $\delta$, a and b are limits respectively of $\chi(z, w)$, $(1-\rho)/(1-r)$ and $(\theta - \varphi)/(1-r)$ then $1 - \delta^2 = 4/(a^{-1} + 2 + a + b^2a^{-1})$.*

PROOF. Clear from the identity

$$1 - \chi^2(z, w) = \frac{(1+r)(1+\rho)}{\left(\dfrac{1-r}{1-\rho} + 2r + r^2\,\dfrac{1-\rho}{1-r} + 2r\rho\,\dfrac{1-\cos(\theta-\varphi)}{(1-r)(1-\rho)}\right)}.$$

We next require some lemmas which later become part of the main theorem (Theorem 2.6).

LEMMA 2.2. *Suppose $f$ is a strictly increasing real function near $0^+$ with $f(0) = f'(0) = 0$. Suppose for some $k_0 \neq 0$, $f(\theta - k_0 f(\theta))/f(\theta) \longrightarrow 1$ as $\theta \longrightarrow 0^+$. Then $f$ is a K-function.*

PROOF. We will assume $k_0 > 0$; the other case is handled similarly. Because $f$ is monotone, the ratio in question tends to 1 if $k_0$ is replaced by $k \in (0, k_0)$. Let $k', \epsilon > 0$ be chosen such that $2\epsilon < k_0$ and $k' - 2\epsilon \equiv k_1 < k_0 < k'$. By our previous remark we may choose $0 < \delta < (k_0 - k_1)/k_1$ and $\theta$ so small that $f(\theta) < (1 + \delta)f(\theta - 2\epsilon f(\theta))$. Then, if $\varphi = \theta - 2\epsilon f(\theta)$,

$$k_1 f(\theta)/f(\varphi) < (1 + \delta)k_1 \equiv \alpha < k_0.$$

Thus,

$$1 \geqslant \frac{f(\theta - k'f(\theta))}{f(\theta)} = \frac{f(\varphi - k_1 f(\theta))}{f(\varphi)} \frac{f(\varphi)}{f(\theta)} \geqslant \frac{f(\varphi - \alpha f(\varphi))}{f(\varphi)} \frac{f(\varphi)}{f(\theta)} \longrightarrow 1.$$

Clearly, the ratio in question then tends to 1 for all $k > 0$.

Now, let $k > 0$, $\varphi = \theta + kf(\theta)$. Then, $k(f(\theta) - f(\varphi)) \leqslant 0$, so $\theta \geqslant \theta + kf(\theta) - kf(\varphi) = \varphi - kf(\varphi)$ and $f(\theta) \geqslant f(\varphi - kf(\varphi))$. Consequently,

$$1 \leqslant \frac{f(\theta + kf(\theta))}{f(\theta)} \leqslant \frac{f(\varphi)}{f(\varphi - kf(\varphi))} \longrightarrow 1$$

by the first part of the proof. The result follows.

LEMMA 2.3. *Let $\{r_n e^{i\theta_n}\}$ be a strict M-sequence in D. Suppose that $k_n = (\theta_n - \theta_{n+1})/(1 - r_n)$ is bounded away from zero and $\gamma_n = (1 - r_{n+1})/(1 - r_n) \longrightarrow 1$. If $f$ is any strictly increasing continuous real function near $0^+$ passing through the points $(\theta_n, 1 - r_n)$, then $f$ is a K-function, and for any two such func-functions $f, g$ we have $f(\theta)/g(\theta) \longrightarrow 1$ as $\theta \longrightarrow 0^+$.*

PROOF. Clearly, $f(0) = f'(0) = 0$. Choose $N$ so large that $\gamma_n > \frac{1}{2}$ for $n \geqslant N$. Select $k_0$ so that $0 < 2k_0 < k_n$ for all $n$. If $\theta_{n+1} \leqslant \theta \leqslant \theta_n$ and $n \geqslant N$, then $k_0 f(\theta) < k_0 f(\theta_n)$. Since $k_0/k_{n+1} < \frac{1}{2} < f(\theta_{n+1})/f(\theta_n)$, we have $\theta - k_0 f(\theta) > \theta_{n+1} - k_{n+1} f(\theta_{n+1}) = \theta_{n+2}$. Consequently,

$$1 \geqslant \frac{f(\theta - k_0 f(\theta))}{f(\theta)} \geqslant \frac{f(\theta_{n+2})}{f(\theta_n)} = \gamma_{n+1} \cdot \gamma_n \longrightarrow 1.$$

By Lemma 2.2, $f$ is a $K$-function. The last assertion of the present lemma follows from the inequalities $f(\theta_{n+1})/f(\theta_n) \leqslant f(\theta)/g(\theta) \leqslant f(\theta_n)/f(\theta_{n+1})$ for $\theta_{n+1} \leqslant \theta \leqslant \theta_n$.

Since we are mainly interested in the homomorphisms in the closure of a $K$-curve the following definition and lemma are natural.

DEFINITION 2.4. Two curves $\Gamma$ and $\Gamma'$ in $D$ terminating at 1 are *equivalent* if $(\Gamma^- - \Gamma) = (\Gamma'^- - \Gamma')$. Two real functions $f$, $g$ defined near $0^+$ are *equivalent* if $f(\theta)/g(\theta) \longrightarrow 1$ as $\theta \longrightarrow 0^+$.

LEMMA 2.5. *Suppose $\Gamma$ is a K-curve with polar representation $1 - r = f(\theta)$ and $\Gamma'$ is a curve in D terminating at 1 which admits a polar representation $1 - r = g(\theta)$ with g continuous, strictly increasing and $g(0) = g'(0) = 0$ for $\theta$ near $0^+$. If g is equivalent to f, then g is a K-function and $\Gamma'$ is equivalent to $\Gamma$. If $\Gamma'$ is equivalent to $\Gamma$, then $\Gamma'$ is a K-curve and g is equivalent to f.*

PROOF. Suppose $g$ is equivalent to $f$. Then, for each $\epsilon > 0$, and small enough $\theta$, $(1 - \epsilon)f(\theta) < g(\theta) < (1 + \epsilon)f(\theta)$. Thus,

$$1 \geqslant \frac{g(\theta - g(\theta))}{g(\theta)} \geqslant \frac{(1 - \epsilon)f(\theta - (1 + \epsilon)f(\theta))}{(1 + \epsilon)f(\theta)}$$

and by Lemma 2.2, since $f$ is a $K$-function, $g$ is also. Next, let $\{r_\alpha e^{i\theta_\alpha}\}$ be a net on $\Gamma$ tending to a homomorphism $h$. Choosing $\{\rho_\alpha e^{i\theta_\alpha}\}$ on $\Gamma'$ we have by direct computation, since $r_\alpha = 1 - f(\theta_\alpha)$ and $\rho_\alpha = 1 - g(\theta_\alpha)$,

$$\chi(r_\alpha e^{i\theta_\alpha}, \rho_\alpha e^{i\theta_\alpha}) = |1 - g(\theta_\alpha)/f(\theta_\alpha)|/|1 + r_\alpha g(\theta_\alpha)/f(\theta_\alpha)|$$

$\longrightarrow 0$. Thus, $\rho_\alpha e^{i\theta_\alpha} \longrightarrow h$. By symmetry, $(\Gamma^- - \Gamma) = (\Gamma'^- - \Gamma')$.

On the other hand, suppose that $\Gamma'$ is equivalent to $\Gamma$. Let $\{\varphi_\alpha\}$ be any universal net tending to $0^+$. Let $\rho_\alpha e^{i\varphi_\alpha}$ be on $\Gamma'$. Since $\Gamma$ is a $K$-curve all the homomorphisms in $\Gamma^-$ are nontrivial [3]. Since $\Gamma'$ is equivalent to $\Gamma$ it follows from [2, Theorem 6.1], that there are corresponding points $r_\alpha e^{i\theta_\alpha}$ on $\Gamma$ with $\chi(r_\alpha e^{i\theta_\alpha}, \rho_\alpha e^{i\varphi_\alpha}) \longrightarrow 0$. By Lemma 2.1, $k_\alpha = (\theta_\alpha - \varphi_\alpha)/(1 - r_\alpha) \longrightarrow 0$ and $(1 - r_\alpha)/(1 - \rho_\alpha) = f(\theta_\alpha)/g(\varphi_\alpha) \longrightarrow 1$. Thus,

$$\frac{f(\varphi_\alpha)}{g(\varphi_\alpha)} = \frac{f(\theta_\alpha - k_\alpha f(\theta_\alpha))}{f(\theta_\alpha)} \frac{f(\theta_\alpha)}{g(\varphi_\alpha)} \longrightarrow 1$$

since $f$ is a $K$-function. Thus $g$ is equivalent to $f$. Consequently, as above, $g$ is a $K$-function so $\Gamma'$ is a $K$-curve.

The next result underlies much of the explicitness obtained in the succeeding analysis of the parts in the closure of $K$-curves.

THEOREM 2.6. *Let $\Gamma$ be an upper tangential curve in D terminating at 1 and let $1 - r = f(\theta)$ be its polar representation with f strictly increasing,* *continuous and* $f(0) = f'(0) = 0$. *Then, each of the following conditions is equivalent to each other condition and to the fact that* $\Gamma$ *is equivalent to a K-curve and f is equivalent to a K-function.*

(1) *There is a function g equivalent to f such that for each sequence* $\{\theta_n\}$, $\theta_n \searrow 0^+$, $[g(\theta_n) - g(\theta_{n+1})]/(\theta_n - \theta_{n+1}) \longrightarrow 0$.

(2) $f(\theta - f(\theta))/f(\theta) \longrightarrow 1$ *as* $\theta \longrightarrow 0^+$.

(3) *For some (or for every)* $k \neq 0$, $f(\theta + kf(\theta))/f(\theta) \longrightarrow 1$ *as* $\theta \longrightarrow 0^+$.

(4) *For some positive bounded measurable function h with essential limit zero as* $\theta \longrightarrow 0^+$, $(1/f(\theta)) \int_0^\theta h(t)\,dt \longrightarrow 1$ *as* $\theta \longrightarrow 0^+$.

(5) *There is a curve* $\Gamma'$ *equivalent to* $\Gamma$ *such that for each strict M-sequence* $\{z_n\}$ *on* $\Gamma'$, *the slopes of the secant lines joining successive points* $z_n$, $z_{n+1}$, *approach* $-\infty$ *as* $n \longrightarrow \infty$.

(6) *For some (or for every)* $k > 0$ *and for some (or for every) M-sequence* $\{r_n e^{i\theta_n}\}$ *on* $\Gamma$ *such that* $(\theta_n - \theta_{n+1})/(1 - r_n) \longrightarrow k$ *one has* $(1 - r_{n+1})/(1 - r_n) \longrightarrow 1$.

(7) *For some (or for every) sequence on* $\Gamma$ *defined recursively by* $\theta_{n+1} = \theta_n - (1 - r_n)$ *one has* $(1 - r_{n+1})/(1 - r_n) \longrightarrow 1$.

(8) *For some (or for every) M-sequence* $\{z_n\}$ *on* $\Gamma$ *such that* $\chi(z_n, z_{n+1}) \longrightarrow \delta \in (0, 1)$ *one has* $(1 - |z_{n+1}|)/(1 - |z_n|) \longrightarrow 1$.

PROOF. We have already seen from Lemma 2.2 that (2) and (3) are equivalent. Suppose (1) holds. Since $g(\theta)/\theta \longrightarrow 0$ we may define a sequence $\{\theta_n\}$, $\theta_n \searrow 0^+$, with $\theta_{n+1} = \theta_n - g(\theta_n)$. (1) then implies that $g(\theta_{n+1})/g(\theta_n) \longrightarrow 1$. It follows immediately from Lemma 2.2 (with $1 - r_n = g(\theta_n)$) that (2) and (3) hold and also hold for the segmental function $f_1$ joining successive points $(\theta_n, g(\theta_n))$ linearly. Furthermore, $f_1(\theta)/f(\theta) \longrightarrow 1$. Let $h = f_1'$, a.e. Clearly, $h$ is an (essentially) positive bounded measurable function with essential limit zero as $\theta \longrightarrow 0^+$ and $f_1$ is recovered by integration. Thus (4) holds. On the other hand, (1) clearly follows from (4) with $g(\theta) = \int_0^\theta h(t)\,dt$ and we see that the first four conditions are equivalent.

Next we show that (1) implies (5). Let $\Gamma'$ correspond to $g$ so $\Gamma'$ is equivalent to $\Gamma$ by Lemma 2.5. Let $\{z_n = r_n e^{i\theta_n}\}$ be a strict $M$-sequence on $\Gamma'$. Suppressing subscripts by letting $r = r_n$, $\rho = r_{n+1}$, $\theta = \theta_n$, $\varphi = \theta_{n+1}$, the slope, $S_n$, of the secant joining $z_n$ to $z_{n+1}$ is

$$\begin{aligned} S_n &= \frac{r\sin\theta - \rho\sin\varphi}{r\cos\theta - \rho\cos\varphi} \\ &= \frac{\sin\varphi - \sin\theta}{\sin(\theta - \varphi)/2}\;\frac{(r-\rho)\sin\theta/(\sin\theta - \sin\varphi) + \rho}{(\rho - r)\cos\varphi/(\sin(\theta-\varphi)/2) + 2r\sin(\theta+\varphi)/2} \\ &\longrightarrow -\infty \quad \text{as } n \longrightarrow \infty \end{aligned}$$

since our assumption implies that $(r-\rho)/(\theta-\varphi) \longrightarrow 0$ (and, of course, $\theta, \varphi \longrightarrow 0$).

Suppose that (5) holds and $\theta_n$ is defined on $\Gamma$ as in (6). If we take $\rho_n e^{i\theta_n}$ on $\Gamma'$, the fact that $\Gamma$ is equivalent to $\Gamma'$ gives $(1+\rho_n)/(1-r_n) \longrightarrow 1$ and $(\theta_n - \theta_{n+1})/(1-\rho_n) \longrightarrow k$. Then with notation similar to the last paragraph, and $\gamma = \gamma_n = (1-r_{n+1})/(1-r_n)$,

$$S_n = -\frac{[2(\cos(\theta+\varphi)/2)\sin(\theta-\varphi)/2]/(1-r) + \gamma\sin\varphi - \sin\theta}{[2(\sin(\theta+\varphi)/2)\sin(\varphi-\theta)/2]/(1-r) + \cos\theta - \gamma\cos\varphi}.$$

If $S_n \longrightarrow -\infty$ as $n \longrightarrow \infty$, it is clear from the fact that $(\theta-\varphi)/(1-r) \longrightarrow k > 0$ that $\gamma_n \longrightarrow 1$ which gives the required conclusion of (6) for every such sequence. *A fortiori* (7) holds.

Suppose, next that for some sequence $\{\theta_n\}$ on $\Gamma$ defined recursively by $\theta_{n+1} = \theta_n - (1-r_n)$ that $(1-r_{n+1})/(1-r_n) \longrightarrow 1$. By Lemma 2.2, $f$ satisfies (3), thus conditions (1) through (7) are equivalent.

Finally, we show that (6) and (8) are equivalent. Suppose (6) is true and $\chi(z_n, z_{n+1}) \longrightarrow \delta \in (0, 1)$ as in (8). Choose any universal net, $n(\alpha)$, of indices, $n(\alpha) \longrightarrow \infty$. Let $h_0 = \lim z_{n(\alpha)}$, $h_1 = \lim z_{n(\alpha)+1}$. Because $\Gamma$ is a $K$-curve, we have all the facts listed in [3]. In particular, there is a unique $\zeta \in D$ such that $L_\alpha(\zeta) = (z_{n(\alpha)} + \zeta)/(1 + \bar{z}_{n(\alpha)}\zeta) \longrightarrow h_1$. Thus, $\chi(L_\alpha(\zeta), z_{n(\alpha)+1}) \longrightarrow 0$ which implies that $(1-|L_\alpha(\zeta)|)/(1-|z_{n(\alpha)+1}|) \longrightarrow 1$. By Lemma 1 of [3],

$$(1-|L_\alpha(\zeta)|)/(1-|z_{n(\alpha)}|) \longrightarrow \operatorname{Re}((1-\zeta)/(1+\zeta)) = a(\zeta).$$

But $a(\zeta) = 1$ since $h_1$ is in the closure of $\Gamma$. Thus, $(1-|z_{n(\alpha)+1}|)/(1-|z_{n(\alpha)}|) \longrightarrow 1$. This being true for every universal net, we have the required result.

Conversely, suppose (8) holds for some $M$-sequence $\{z_n\}$. Since $\chi(z_n, z_{n+1}) \longrightarrow \delta \in (0, 1)$ and $1-|z_{n+1}|/1-|z_n| \longrightarrow 1$, we must have, by Lemma 2.1, that $(\theta_n - \theta_{n+1})/(1-r_n) \longrightarrow 2\delta/\sqrt{1-\delta^2}$ and (6) holds.

The next result gives very "explicit" descriptions of the Wermer maps onto parts in the closure of a $K$-curve.

THEOREM 2.7. *Let $\Gamma$ be a K-curve with polar representation $1-r=f(\theta)$ and let $b_0 > 0$. Choose $0 < \theta_1 < \pi/2$ and define $z_n = r_n e^{i\theta_n}$ recursively on $\Gamma$ by $\theta_{n+1} = \theta_n - b_0 f(\theta_n)$. Let $h_0$ be a homomorphism in the closure in $\mathcal{D}_1$ of $\{z_n\}$, say, $h_0 = \lim z_{n(\alpha)}$. Let $\tau$ be the Wermer map of D onto the part of $h_0$ such that $\tau(0) = h_0$. Given $z_0 \in D$, let N be the greatest integer in $b(z_0)/b_0$ and let $\lambda = b(z_0) - Nb_0$. Then $\tau(z_0) = \lim \rho_{n(\alpha)} e^{i\varphi_{n(\alpha)}}$ where $\varphi_{n(\alpha)} = \theta_{n(\alpha)+N} - \lambda f(\theta_{n(\alpha)+N})$ and $\rho_{n(\alpha)} = 1 - f(\varphi_{n(\alpha)}) a(z_0)$,*

PROOF. We will give the proof for $b(z_0) > 0$. The proof for negative values is similar.

Let $\varphi'_n = \theta_n - b(z_0)f(\theta_n)$, $\rho'_n = 1 - f(\varphi'_n)a(z_0)$. Then, from [3], $\tau(z_0) = \lim \rho'_{n(\alpha)}e^{i\varphi'_{n(\alpha)}}$ so the theorem will follow if we can prove that

$$\chi(\rho_{n(\alpha)}e^{i\varphi_{n(\alpha)}}, \rho'_{n(\alpha)}e^{i\varphi'_{n(\alpha)}}) \longrightarrow 0.$$

By Lemma 2.1 this will follow from proving that $(\varphi_n - \varphi'_n)/(1-\rho_n) \longrightarrow 0$ and $(1-\rho_n)/(1-\rho'_n) \longrightarrow 1$.

We estimate first that

$$\begin{aligned}\varphi'_n &= \theta_n - \sum_{k=0}^{N-1} b_0 f(\theta_n) - \lambda f(\theta_n)\\ &\leqslant \theta_n - \sum_{k=0}^{N-1} b_0 f(\theta_{n+k}) - \lambda f(\theta_{n+N})\\ &= \theta_{n+N} - \lambda f(\theta_{n+N}) = \varphi_n.\end{aligned}$$

Let $\epsilon > 0$. By Theorem 2.6, condition (7) for large $n, f(\theta_{n+j}) \leqslant (1+\epsilon)f(\theta_{n+k})$, $j, k = 0, \ldots, N$. We then estimate that

$$\begin{aligned}\varphi'_n &\geqslant \theta_n - \sum_{k=0}^{N-1} b_0(1+\epsilon)f(\theta_{n+k}) - \gamma(1+\epsilon)f(\theta_{n+N})\\ &= \varphi_n - \epsilon b_0 \sum_{k=0}^{N-1} f(\theta_{n+k}) - \epsilon\lambda f(\theta_{n+N})\\ &\geqslant \varphi_n - \epsilon[b_0(1+\epsilon)N + \lambda]f(\theta_{n+N}).\end{aligned}$$

Therefore,

$$0 \leqslant \frac{\varphi_n - \varphi'_n}{1-\rho_n} \leqslant \frac{\epsilon(b(z_0) + \epsilon b_0 N)}{a(z_0)} \frac{f(\theta_{n+N})}{f(\theta_n)}.$$

It is clear that $(\varphi_n - \varphi'_n)/(1-\rho_n) \longrightarrow 0$. Similarly,

$$(1-\rho_n)/(1-\rho'_n) = f(\theta_n)/f(\varphi'_n) \leqslant f(\theta_n)/f(\theta_{n+N+1}) \longrightarrow 1.$$

Using [3] it is not hard to see that another way of describing geometrically what we have done in Theorem 2.7 is the following: For each $\gamma \in (0, \infty)$ and each $\delta \in (|\gamma - 1|/|\gamma + 1|, 1)$, the circle $\chi(z, z_n) = \delta$ eventually intersects the curve $\Gamma(f, \gamma)$ (given by $f(\theta) = \gamma(1-r)$) in two points which are asymptotically equivalent to the points on $\Gamma(f, \gamma)$ with arguments $\theta_{n+N} - \lambda f(\theta_{n+N})$ and $\theta_{n-N} + \lambda f(\theta_{n-N})$ where $N$ and $\lambda$ are computed by $|b| = Nb_0 + \lambda$, $0 \leqslant \lambda < b_0$, and

$$b^2 = [\delta^2 - ((\gamma-1)/(\gamma+1))^2]\,[(\gamma+1)^2/\gamma^2(1-\delta^2)].$$

(This evaluation of $b$ follows from Lemma 2.1 and the relation $\gamma = 1/a$ from [3].)

We close this section with an example which demonstrates that the class of $K$-functions is definitely wider than the class considered in [3].

EXAMPLE 2.8. There exists a $K$-function which is equivalent to no convex function.

PROOF. First, suppose $h$ is any $K$-function whose graph is piecewise linear. Suppose $g$ is convex and for some $K > 0$, $K^{-1} \leqslant h(\theta)/g(\theta) \leqslant K$ for $\theta$ near $0^+$. Let $f$ be the convex function whose graph forms the lower boundary of the convex hull of the graph of $h$. Let $\{\theta_n\}$ be the sequence with $\theta_n \searrow 0$ such that $(\theta_n, h(\theta_n))$ is a vertex lying on the graph of $f$. Let $\theta_{n+1} \leqslant \theta \leqslant \theta_n$, $\theta = \lambda\theta_n + (1-\lambda)\theta_{n+1}$. Then

$$K^{-1}f(\theta) \leqslant f(\theta)g(\theta)/h(\theta) \leqslant g(\theta) \leqslant \lambda g(\theta_n) + (1-\lambda)g(\theta_{n+1})$$
$$\leqslant K[\lambda h(\theta_n) + (1-\lambda)h(\theta_{n+1})] = Kf(\theta).$$

Thus, if there is such a $g$, the function $f$ has the same properties.

Define $f$ to be the piecewise linear function passing through the points $(1/n, 1/n!)$. Define $h$ to be the piecewise linear function joining these points and the additional points $((n^2+1)/n^2(n+1), 1/2 \cdot n!)$. It is not hard to calculate that $f$ is the convex function bearing the same relationship to $h$ as above. Furthermore, the secant slopes of $h$ tend to zero so that by Theorem 2.6, condition (1), $h$ is a $K$-function. But, if $\theta_n = (n^2+1)/n^2(n+1)$, then $h(\theta_n)/f(\theta_n) \longrightarrow \infty$. Combining this with the above initial remarks verifies the example.

In fact, it is not hard to see that for each given convex function $g$, the $K$-curve corresponding to the function $h$ of the example contains whole parts in its closure which are not in the closure of the curve defined by $g$.

3. **Interpolating sequences and Blaschke products.** We begin with some facts about $M$-sequences. The first result can be computed by brute force and was noted in [4], while the second follows at once.

LEMMA 3.1. *If* $z = re^{i\theta}$, $w_1 = \rho_1 e^{i\varphi_1}$, $w_2 = \rho_2 e^{i\varphi_2}$ *and* $0 < r \leqslant \rho_1 \leqslant \rho_2 < 1$, $0 < \varphi_2 \leqslant \varphi_1 \leqslant \theta < \pi/2$, *then* $\chi(z, w_1) \leqslant \chi(z, w_2)$.

COROLLARY 3.2. *Suppose* $\{z_n\}$ *is an M-sequence and for some* $\delta > 0$, $\chi(z_n, z_{n+1}) \geqslant \delta$. *Then* $\chi(z_n, z_m) \geqslant \delta$ *for all n, m.*

The next result is basic to the section and allows us to use some of the techniques of [4] to improve Wortman's theorem on interpolating sequences.

LEMMA 3.3. *Let* $\{z_n = r_n e^{i\theta_n}\}$ *be an M-sequence such that* $\chi(z_n, z_{n+1}) \geqslant \delta > 0$ *for some* $\delta$. *Then,* $\{z_n\}$ *can be partitioned into two M-sequences* $\{\alpha_k\}$ *and* $\{\beta_k = \rho_k e^{i\varphi_k}\}$ *such that:*

(1) $\{\alpha_k\}$ *is geometric, i.e.,* $1 - |\alpha_{k+1}|/1 - |\alpha_k| \leqslant c < 1$ *for some c.*

(2) *The numbers* $(\varphi_k - \varphi_{k+1})/(1-\rho_k)$ *are bounded away from zero.*

PROOF. Let

$$a_n = (1 - r_{n+1})/(1-r_n), \qquad b_n^2 = 2(1-\cos(\theta_n - \theta_{n+1}))/(1-r_n)^2,$$

and for $0 < c < 1$, let $N_c = \{n: a_n > c\}$. If $N_c$ is always eventually empty we may take $\alpha_k = z_k$ and $\{\beta_k\}$ to be empty. Otherwise, we compute, suppressing indices with $r_n = r$, $r_{n+1} = \rho$,

$$\frac{(1+r)(1+\rho)}{a^{-1} + 2r + r^2 a + r\rho b^2 a^{-1}} = 1 - \chi^2(z_n, z_{n+1}) \leqslant 1 - \delta^2 < 1.$$

Solving for $b_n^2$ and taking $a$ as any limit point of $\{a_n\}$ through $N_c$,

$$\liminf b_n^2 \geqslant \frac{(1+a)^2\delta^2 - (1-a)^2}{(1-\delta^2)} \geqslant \frac{(1+c)^2\delta^2 - (1-c)^2}{(1-\delta^2)},$$

since for $n \in N_c$, $a_n > c$ so $a \geqslant c$. Therefore, we may choose $b_0 > 0$ and $c$ so close to 1 that $b_n > b_0$ for $n \in N_c$.

Fix such a $c$. Let $\{\alpha_k\}$ be the sequence (in natural order) $\{z_n: n \notin N_c\}$. Given $k$, for some $n \notin N_c$ and $m > n$, $\alpha_k = z_n$, $\alpha_{k+1} = z_m$. Then

$$1 - |\alpha_{k+1}|/1 - |\alpha_k| = (1-r_m)/(1-r_n) \leqslant (1-r_{n+1})/(1-r_n) \leqslant c.$$

Thus $\{\alpha_k\}$ is geometric.

Consider the remaining sequence $\{z_n: n \in N_c\} = \{\beta_k\}$ in natural order. Given $k$, for some $n \in N_c$ and $m > n$, $\beta_k = z_n$ and $\beta_{k+1} = z_m$. Then, because of the choice of $c$,

$$\begin{aligned}(\varphi_k - \varphi_{k+1})/(1-\rho_k) &= (\theta_n - \theta_m)/(1-r_n) \\ &\geqslant (\theta_n - \theta_{n+1})/(1-r_n) = b_n > b_0 > 0.\end{aligned}$$

A crucial observation of Wortman [4] for us is that one can make an excellent lower estimate of an infinite real product $\Pi\, a_n$ provided the numbers $\{a_n\}$ are bounded away from zero on the left.

LEMMA 3.4. *Let* $\{a_n\}$ *be a real sequence such that for some* $\delta > 0$, $\delta \leqslant a_n < 1$ *for all n. Then,*

$$\exp\left(-\delta^{-1}\sum_{k=1}^{\infty}(1-a_k)\right) \leqslant \prod_{k=1}^{\infty} a_k \leqslant \exp\left(-\sum_{k=1}^{\infty}(1-a_k)\right).$$

PROOF. The proof is standard beginning with the inequality $(1-x) < -\log x < \delta^{-1}(1-x)$ for $\delta \leqslant x \leqslant 1$.

We now isolate as a lemma the computations required for both of the main results, Theorems 3.6 and 3.7, of this section.

LEMMA 3.5. *Let* $\{z_k = r_k e^{i\theta_k}\}$ *be an M-sequence in D such that* $(\theta_k - \theta_{k+1})/(1 - r_k) \geqslant b_0 > 0$ *for all k. Assume further that for all k,* $r_k > 1/2$ *and* $0 < \theta_k < \pi/3$. *Let* $w = \rho e^{i\varphi} \in D$ *and let n be the index for which* $\theta_{n+1} \leqslant \varphi \leqslant \theta_n$. *Then*

$$\Sigma = \sum_{\substack{k=1 \\ k \neq (n-1),\dots,(n+2)}}^{\infty} 1 - \chi^2(w, z_k)$$

$$\leqslant \frac{8}{b_0\sqrt{\rho}}\left[\tan^{-1}\frac{2(1-\rho)}{b_0\sqrt{\rho}\,(1-r_{n-1})} + \tan^{-1}\frac{2(1-\rho)}{b_0\sqrt{\rho}\,(1-r_{n+1})}\right].$$

PROOF. The condition $0 < \theta_k < \pi/3$ implies that $1 - \cos(\theta_k - \varphi) \geqslant (\theta_k - \varphi)^2/4$. Thus, using $r_k > ½$ and beginning with the expression for $1 - \chi^2$ listed at the beginning of §2, one has

$$1 - \chi^2(w, z_k) \leqslant 4\,\frac{(1-r_k)(1-\rho)}{(1-\rho)^2 + \rho(\theta_k - \varphi)^2/4}\,.$$

Replacing each term of $\Sigma$ by this estimate and using the facts that for $k = 1, \dots, n-2$ one has $1 - r_k \leqslant (\theta_k - \theta_{k+1})/b_0$, and for $k = n+3, \dots$ one has $1 - r_k \leqslant 1 - r_{k-1} \leqslant (\theta_{k-1} - \theta_k)/b_0$, we estimate

$$\Sigma \leqslant \frac{4}{b_0}\left[\sum_{k=1}^{n-2}\frac{(1-\rho)(\theta_k - \theta_{k+1})}{(1-\rho)^2 + \rho(\theta_k - \varphi)^2/4} + \sum_{k=n+3}^{\infty}\frac{(1-\rho)(\theta_{k-1} - \theta_k)}{(1-\rho)^2 + \rho(\varphi - \theta_k)^2/4}\right]$$

$$\leqslant \frac{8}{b_0\sqrt{\rho}}\left[\left(\int_A^B + \int_C^D\right)\frac{d\theta}{1+\theta^2}\right]$$

where $A = \sqrt{\rho}\,(\theta_{n-1} - \varphi)/2(1-\rho)$, $B = \sqrt{\rho}\,(\theta_1 - \varphi)/2(1-\rho)$, $C = \sqrt{\rho}\,(\varphi - \theta_{n+2})/2(1-\rho)$, $D = \sqrt{\rho}\,\varphi/2(1-\rho)$. This last step follows from standard type estimates of sums by integrals. The inequality remains if one replaces $B$ and $D$ by infinity and, by the choice of $n$, if one replaces $A$ and $C$, respectively, by the smaller values

$$A \geqslant \sqrt{\rho}\,(\theta_{n-1} - \theta_n)/2(1-\rho) \geqslant b_0\sqrt{\rho}\,(1 - r_{n-1})/2(1-\rho)$$

and

$$C \geqslant \sqrt{\rho}\,(\theta_{n+1} - \theta_{n+2})/2(1-\rho) \geqslant b_0\sqrt{\rho}\,(1 - r_{n+1})/2(1-\rho).$$

The result then follows.

We recall Carleson's condition (see [1]) that a sequence $\{z_n\}$ in $D$ is an $H^{\infty}$-interpolating sequence if and only if $\Pi_{k\neq n}\chi(z_k, z_n) \geqslant \delta > 0$ for some $\delta$. Notwithstanding the elegance of this characterization it is generally not an easy matter to check the condition in special cases. In [4] Wortman verified Carleson's condition under the hypothesis that the points $\{z_n\}$ form a strict $M$-sequence lying on a convex curve terminating at 1 such that $\chi(z_n, z_{n+1}) \geqslant \delta_0 > 0$ for all $n$. We show next that the convex curve is superfluous.

THEOREM 3.6. *Let $\{z_n\}$ be a strict M-sequence in D. Then, a necessary and sufficient condition for $\{z_n\}$ to be an $H^{\infty}$-interpolating sequence is that $\chi(z_n, z_{n+1}) \geqslant \delta_0 > 0$ for some $\delta_0$.*

PROOF. That $\chi(z_n, z_{n+1}) \geqslant \delta_0 > 0$ if $\{z_n\}$ is interpolating follows at once from Carleson's condition.

Suppose conversely that $\chi(z_n, z_{n+1}) \geqslant \delta_0 > 0$. By Lemma 3.3, $\{z_n\}$ can be partitioned into a geometric sequence $\{\alpha_k\}$ and an $M$-sequence $\{\beta_k = \rho_k e^{i\varphi_k}\}$ such that $(\varphi_k - \varphi_{k+1})/(1-\rho_k) \geqslant b_0 > 0$ for some $b_0$. Since our result does not depend on the first few terms we may assume $\rho_k > 1/2$ and $0 < \theta_k < \pi/3$. Then Lemma 3.5 applies with $\{z_k\} = \{\beta_k : k \neq n\}$ and $w = \beta_n$. Then certainly $\Sigma_{k\neq n} 1 - \chi^2(\beta_n, \beta_k) \leqslant M$ for some constant $M$. Since by Corollary 3.2 one has $\chi(\beta_n, \beta_k) \geqslant \delta_0$ for all $n$, $k$, Carleson's condition for $\{\beta_k\}$ follows from Lemma 3.4. Thus $\{\beta_k\}$ is interpolating. It is well known that any geometric sequence, and thus $\{\alpha_k\}$, is interpolating.

Let $B_1, B_2$ be the Blaschke products with zeros $\{\alpha_k\}$ and $\{\beta_k\}$, respectively. Then $B = B_1 B_2$ is the Blaschke product with zeros $\{z_n\}$. Referring to the very last material concerning interpolating sequences in [1] we have that the closure in $\mathcal{D}$ of each of $\{\alpha_k\}$ and $\{\beta_k\}$ is homeomorphic to the Stone-Čech compactification of the integers and $B_1$ and $B_2$ vanish only in the closures of $\{\alpha_k\}$ and $\{\beta_k\}$, respectively. But applying Corollary 3.2 to the sequence $\{z_n\}$, we see that $\{\alpha_k\}$ and $\{\beta_k\}$ have disjoint closures. Thus $\{z_n\}^-$ is homeomorphic to the Stone-Čech compactification of the integers and $B$ vanishes only in the closure of $\{z_n\}$. Consequently, $\{z_n\}$ is interpolating.

In the special case that the sequence $\{z_n\}$ is asymptotically equally spaced and lies on a $K$-curve one can estimate the boundary behavior of the modulus of the Blaschke product with zeros $\{z_n\}$.

THEOREM 3.7. *Let $\Gamma$ be a K-curve with polar representation $1 - r = f(\theta)$. Let $\{z_k = r_k e^{i\theta_k}\}$ be points chosen on $\Gamma$ recursively by $\theta_{k+1} = \theta_k - b_0 f(\theta_k)$ for some $b_0 > 0$. Let B be the Blaschke product with zeros $\{z_k\}$. Then*:

(1) *For $\gamma \in (0, \infty)$, $\gamma \neq 1$,*

$$\liminf \{|B(w)|: w = \rho e^{i\varphi} \longrightarrow 1, f(\varphi) = \gamma(1-\rho)\} \geqslant \exp(-(2\delta)^{-1}M(\gamma))$$

*where* $\delta = (|\gamma - 1|/|\gamma + 1|)^2$ *and* $M(\gamma) = (16/b_0)\tan^{-1}(4/b_0\gamma) + 4(1-\delta)$.

(2) *Let* $h_0$ *be in the closure in* $\mathcal{D}_1$ *of* $\{z_k\}$ *and let* $\tau$ *be the Wermer map of* $D$ *onto the part of* $h_0$ *such that* $\tau(0) = h_0$. *Then* $B \circ \tau$ *is the Blaschke product whose zeros have coordinates* $a(z) = 1$, $b(z) = 0, \pm b_0, \pm 2b_0, \ldots$ .

PROOF. Let $1 \neq \gamma > 0$. Every homomorphism in $\mathcal{D}_1$ in the closure of $\Gamma(f, \gamma) = \{\rho e^{i\varphi}: f(\varphi) = \gamma(1-\rho)\}$ is in the same part as some homomorphism in the closure in $\mathcal{D}_1$ of $\{z_k\}$ since the points $\{z_k\}$ are asymptotically equally spaced (by $b_0/\sqrt{4 + b_0^2}$). Let $h_0$ be in the closure of $\{z_k\}$, say, $h_0 = \lim z_{n(\alpha)}$. Let $\tau$ be the Wermer map of $D$ onto the part of $h_0$ such that $\tau(0) = h_0$. Let $z_0$ be a point in $D$ such that $a(z_0) = 1/\gamma$ and let $N$ be the greatest integer in $b(z_0)/b_0$ and $\lambda = b(z_0) - Nb_0$. By Theorem 2.7, $\tau(z_0) = \lim \rho_{n(\alpha)}e^{i\varphi_{n(\alpha)}}$ where $\varphi_{n(\alpha)} = \theta_{n(\alpha)+N} - \lambda f(\theta_{n(\alpha)+N})$ and $1 - \rho_{n(\alpha)} = f(\varphi_{n(\alpha)})/\gamma$.

We may apply Lemma 3.5 to $\{z_k\}$ and $w = \rho_{n(\alpha)}e^{i\varphi_{n(\alpha)}}$. From the above we see that for $b(z_0) \geqslant 0$, $\theta_{n(\alpha)+N+1} \leqslant \varphi_{n(\alpha)} \leqslant \theta_{n(\alpha)+N}$ while for $b(z_0) < 0$, $\theta_{n(\alpha)+N} \leqslant \varphi_{n(\alpha)} \leqslant \theta_{n(\alpha)+N-1}$. We will handle explicitly only the case where $b(z_0) \geqslant 0$ since as we shall see the only important ingredient in our argument will be that the index of $\theta_{n(\alpha)}$ is advanced or retarded by a fixed amount.

To apply Lemma 3.5, we first notice that

$$\frac{1 - \rho_{n(\alpha)}}{1 - r_{n(\alpha)+N+1}} = \frac{\gamma^{-1}f(\theta_{n(\alpha)+N}) - \lambda f(\theta_{n(\alpha)+N})}{f(\theta_{n(\alpha)+N+1})} \leqslant 2\gamma^{-1}$$

for $n(\alpha)$ large since $f$ is a $K$-function and $N$ is *fixed*. Similarly,

$$\frac{1 - \rho_{n(\alpha)}}{1 - r_{n(\alpha)+N}} \leqslant 2\gamma^{-1}.$$

Next, from [3] we have $\chi(w, z_k) \geqslant |\gamma - 1|/|\gamma + 1|$. Combining all of this we have

$$\limsup_{\alpha} \sum_{k=1}^{\infty} 1 - \chi^2(w_{n(\alpha)}, z_k) \leqslant \frac{16}{b_0}\tan^{-1}\frac{4}{\gamma b_0} + 4\left(1 - \left(\frac{|\gamma - 1|}{|\gamma + 1|}\right)^2\right).$$

If we apply Lemma 3.4 to $|B|^2$ and recall the remark at the beginning of the proof we see that (1) follows.

In fact, the estimate in (1) holds for $|(B \circ \tau)(z_0)|$ whenever $a(z_0) = 1/\gamma$ regardless of the value of $b(z_0)$. If we fix $z$ to lie on any curve $0 \neq b(z) = b =$ constant, $-\infty < b < \infty$, and let $\gamma \longrightarrow \infty$ we have $|(B \circ \tau)(z)| \longrightarrow 1$. That is, the radial limits of $|B \circ \tau|$ are 1 at all points of the unit circle except $-1$. Taking $b(z) = 0$ and letting $\gamma \longrightarrow 0$ we see that $|B \circ \tau|$ does not tend to zero radially at $-1$. Thus $B \circ \tau$ is an inner function with no zero radial limits. Hence $B \circ \tau$ is a Blaschke product with a singularity only at $-1$. Since from Theorem 3.6, $\{z_k\}$ is interpolating, $B \circ \tau$ vanishes exactly at those points in the part of $h_0$ which are in the closure of $\{z_k\}$. From Theorem 2.7 these points are precisely those given in (2).

COROLLARY 3.8. *The closure in $\mathcal{D}_1$ of any K-curve disconnects $\mathcal{D}_1$.*

PROOF. Let $\Gamma$ be a $K$-curve with polar representation $1 - r = f(\theta)$. Let $B$ be one of the Blaschke products discussed in Theorem 3.7. For $0 \leqslant \gamma \leqslant \infty$, let $A(f, \gamma) = \{h$: for some universal net $r_\alpha e^{i\theta_\alpha} \longrightarrow h, f(\theta_\alpha)/(1-r_\alpha) \rightarrow \gamma\}$. From [3] one sees that if $\gamma' \in (0, \infty)$ and $\gamma \neq \gamma'$, then $A(f, \gamma) \cap A(f, \gamma') = \varnothing$. Since $|B| = 1$ on $A(f, \infty)$ and $|B| < 1$ on $A(f, 0)$ we see that $A(f, 0) \cap A(f, \infty)$ is also empty. It is easily seen that $\bigcup \{A(f, \gamma): \gamma \leqslant 1\}$ is closed and that its complement is $A_1 = \bigcup \{A(f, \gamma): \gamma > 1\}$. Similarly, $A_2 = \bigcup \{A(f, \gamma): \gamma < 1\}$ is open and $A_1 \cap A_2 = \varnothing$. Since $\mathcal{D}_1 - A(f, 1) = A_1 \cup A_2$ the result is clear.

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